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The Exterior Dirichlet Problem for a Strongly Nonlinear Elliptic Equation

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Let Ω be the exterior of a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$. For a possibly nonlinear elliptic operator A in divergence form and a differentialbe function $p: \mathbb{R} \to \mathbb{R}$ with $p(0)=0$, $p'(t) \geqslant 0$ $\forall t \in \mathbb{R}$ we discuss the solvability of the boundary value problem

$$Au + p(u) = f - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \qquad \text{in } \Omega,\ u=0 \text{ on } \partial\Omega$$

under various restrictive conditions on $p(\cdot)$ which, however, would still allow more or less liberal exponential growths. The right-hand side of the equation belongs to some subspaces of the dual of $W_0^{1,2}(\Omega)$.

I. INTRODUCTION

Let D be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ and $\Omega = \mathbb{R}^N - \bar{D}$. We assume that the boundary $\partial\Omega$ of Ω is smooth. Let $p: \mathbb{R} \to \mathbb{R}$ be a differentiable function. We discuss the solvability of the boundary value problem (abbreviated to BVP in the sequel):

$$Au + p(u) = f - D_i f_i \qquad \text{in} \quad \Omega \tag{1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega \tag{2}$$

where the A is an elliptic differential operator in divergence form, possibly nonlinear. In Theorem I below we prove that if A is of the form

$$A = -D_j[a_{ij}(x)\, D_i u], \qquad a_{ij}(\cdot) \in L^\infty(\Omega) \qquad (i, j = 1,..., N),$$

and

$$p'(t) \geqslant \lambda > 0 \qquad \forall t \in \mathbb{R},$$

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then the BVP (1), (2) has a unique solution in $W^{1,2}(\Omega)$ in some weak sense for any $f \in L^2(\Omega) \cap L^r_{\text{loc}}(\Omega)$ with $r > N/2$ and $f_i \in L^2(\Omega) \cap L^s_{\text{loc}}(\Omega)$ with $s > N$ ($i = 1,..., N$). We wish to point out that if $p(\cdot)$ satisfies an additional restrictive condition that essentially excludes functions with exponential growth then it has been proved in [1, Theorem 3.1], that with $A = -\Delta$, the BVP (1), (2) is solvable in a somewhat weaker sense for any f, $f_i \in L^2(\Omega)$ ($i = 1,..., N$). Furthermore, using the method of lower and upper solutions [5, 6] we have proved in [6, Theorem 5] the special case of Theorem I when $f_1 = \cdots = f_N = 0$. As usual with the method of lower and upper solutions, we also have some extra information about the "size" of the solution obtained by that method. However, we were unable to construct a lower solution ϕ and an upper solution ψ with $\phi \leqslant \psi$ of the BVP (1), (2) if the terms f_i ($i = 1,..., N$) are present on the right-hand side of (1).

Perhaps it is appropriate to mention that besides the connection described above, the work presented in this paper makes use of no results from [5, 6].

In Theorem II whose proof is considerably more complicated than that of Theorem I we admit nonlinear operators A which are essentially of Leray–Lions type. To establish solvability for these operators we have to put further restrictions on the function f_i, namely $f_i \in L^4(\Omega)$ ($i = 1,..., N$) and on the behavior of $p(\cdot)$ which still allows some exponential growths. As additional compensation, the BVP (1), (2) is then solvable in a stronger sense than in Theorem 1. Some variations of Theorem II are given in Theorems III and IV.

In proving these theorems, apart from the difficulties normally associated with an unbounded domain, a major obstacle arises from the possibly unlimited growth of the function $p(\cdot)$. We shall make essential use of a result we prove earlier [4] on the existence of bounded solutions of stronlgly nonlinear elliptic equations on bounded domains as well as interior L^∞ estimates [8, Chap. 3, Sect. 13]. In the case of Theorem II we also employ Browder's result [3] on the pseudo-monoticity of operators of Leray–Lions type defined on unbounded domains.

Theorem V concerns the situation when we only have $p'(t) \geqslant 0$ $\forall t \in \mathbb{R}$: it is then proved that the BVP has a solution in some weighted Sobolev's space.

Finally we note that strongly nonlinear elliptic equations on unbounded domains are considered, among others, by Brézis and Browder in [2] and Webb in [12] under different hypotheses on the strongly nonlinear term. Their results are different from ours (Please see Remark II after Theorem II further down for a more detailed discussion.)

II. MAIN RESULTS

Suppose that $a_{ij}(\cdot) \in L^\infty(\Omega)$ $(i, j = 1, \ldots, N)$ and there exists $\nu > 0$ such that

$$a_{ij}(x)\,\xi_i\xi_j \geqslant \nu|\xi|^2 \qquad \text{for} \quad \forall \xi \in \mathbb{R}^N, \text{ a.a. } x \in \Omega,$$

with the usual convention that if an index is repeated then summation from 1 to N over that index is implied. Consider the BVP:

$$-D_j[a_{ij}(x)\,D_i u] + p(u) = f - D_i f_i \qquad \text{in} \quad \Omega, \tag{3}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{4}$$

where $D_i = \partial/\partial x_i$ $(i = 1, \ldots, N)$. We have

THEOREM I. *Suppose that*

(P1) $p(0) = 0$ *and there exists* $\lambda > 0$ *such that* $p'(t) \geqslant \lambda > 0$ $\forall t \in \mathbb{R}$. *Then for any* $f \in L^2(\Omega) \cap L^r_{\text{loc}}(\Omega)$ *with* $r > N/2$ *and* $f_i \in L^2(\Omega) \cap L^s_{\text{loc}}(\Omega)$ *with* $s > N$ $(i = 1, \ldots, N)$ *the* BVP (3), (4) *has a unique solution* $u \in W_0^{1,2}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ *in the sense that for every* $v \in W^{1,2}(\Omega)$ *with compact support we have*

$$\int_\Omega \{a_{ij}(x)\,D_i u D_j v + p(u)\,v\}\,dx = \int_\Omega (fv + f_i D_i v)\,dx \tag{5}$$

Before proving this theorem we wish to make a remark delineating the relationship between our result and previous ones on the BVP (3), (4).

Remark I. It is proved in [1, Theorem 3.1] that if $p(\cdot)$ satisfies (P1) as well as the condition

(P2) $|p(\mu t)| \leqslant \gamma(\mu)\,|p(t)|$ $\forall \mu, t \in \mathbb{R}$,

then the BVP (3), (4) has a solution $u \in W_0^{1,2}(\Omega)$ for any $f, f_i \in L^2(\Omega)$ $(i = 1, \ldots, N)$ in the sense that (5) is valid for every $v \in C_0^\infty(\Omega)$; thus this solution is of a somewhat weaker sense than the solution obtained via Theorem I above. Furthermore condition (P2) excludes functions like

$$p(t) = e^t - 1 \quad \text{if } t \geqslant 0, \; p(t) = -e^{-t} + 1 \quad \text{if } t < 0$$

which satisfies (P1).

Proof of Theorem I. For $r > 0$ let B_r be the open ball centered at 0 and of radius r in $\mathbb{R}^N$ and let $\Omega_r = \Omega \cap B_r$. Let k_0 be a fixed integer such that $\mathbb{R}^N - \Omega \subset B_{k_0}$. For each integer $k > k_0$ let

$$\zeta_k(x)=1 \qquad \text{if} \quad |x|\leqslant k,$$

$$\zeta_k(x)=\exp\left\{\frac{1}{(|x|-k)^2-1}\right\} \qquad \text{if} \quad k<|x|<k+1,$$

$$\zeta_k(x)=0 \qquad \text{if} \quad |x|\geqslant k+1.$$

We note that $\zeta_k(\cdot)\in C_0^1(\Omega)$.

Proof of Uniqueness. Suppose that the BVP (1), (2) has two solutions u_1 and u_2. Let $w=u_2-u_1$, $w^+(x)=\max[w(x), 0]$, $w^-(x)=w^+(x)-w(x)$. From (5) with $v=\zeta_k w^+$ $(k>k_0)$ we obtain

$$\int_\Omega \{a_{ij}D_i w^+ D_j(\zeta_k w^+)+[p(u_2)-p(u_1)]\,\zeta_k w^+\}\,dx=0. \tag{6}$$

Since $p'(t)\geqslant\lambda>0$ $\forall t\in\mathbb{R}$, we have

$$[p(u_2)-p(u_1)]\,\zeta_k w^+\geqslant 0.$$

Therefore we deduce from (6) that

$$0\leqslant\int_\Omega \zeta_k a_{ij}D_i w^+ D_j w^+\,dx\leqslant -\int_\Omega a_{ij}\cdot D_i w^+\cdot D_j\zeta_k\cdot w^+\,dx. \tag{7}$$

By our construction of ζ_k, $|||\text{grad}\,\zeta_k(\cdot)|||_{L^\infty(\Omega)}$ is bounded for $k>k_0$ and $\text{grad}\,\zeta_k(x)\neq 0$ only when $k<|x|<k+1$. Furthermore, since $a_{ij}(\cdot)\in L^\infty(\Omega)$ $(i, j=1,..., N)$ and $w\in W^{1,2}(\Omega)$, letting $k\to\infty$ we see that the integral on the right-hand side of (7) tends to 0. Thus

$$\int_\Omega a_{ij}D_i w^+ D_j w^+\,dx=0$$

and it follows from this that $|\text{grad}\,w^+|=0$. Then (6) gives

$$\int_\Omega [p(u_2)-p(u_1)]\,\zeta_k w^+\,dx=0$$

and we conclude that $w^+=0$. Similarly $w^-=0$.

Proof of Existence. The difficulty in proving existence arises from the possibly unlimited growth of $p(\cdot)$. For each integer $n>k_0$ consider the BVP

$$-D_j[a_{ij}(x)\,D_i u]+p(u)=f-D_i f_i \qquad \text{in} \quad \Omega_n \tag{8}$$

$$u=0 \qquad \text{on} \quad \partial\Omega_n. \tag{9}$$

Since $f \in L^r(\Omega_n)$ with $r > N/2$, $r \geqslant 2$, $f_i \in L^s(\Omega_n)$ $(i = 1,..., N)$ with $s > N \geqslant 2$, by the result of [4] the BVP (8), (9) has a solution $u_n \in W^{1,2}(\Omega_n) \cap L^\infty(\Omega_n)$ in the sense that for every $v \in W_0^{1,2}(\Omega_n)$:

$$\int_{\Omega_n} \{a_{ij} D_i u_n D_j v + p(u_n)\, v\}\, dx = \int_{\Omega_n} \{fv + f_i D_i v\}\, dx. \tag{10}$$

We extend u_n to the whole of Ω by defining $u_n(x) \equiv 0$ for $x \in \Omega - \Omega_n$ and, for convenience, still denote by u_n this function of $W_0^{1,2}(\Omega)$. We have

$$p(t)\, t \geqslant \lambda t^2 \qquad \forall t \in \mathbb{R} \tag{11}$$

because $p'(t) \geqslant \lambda > 0$. Therefore it follows from (10) with $v = u_n$ that

$$\|u_n\|_{W_0^{1,2}(\Omega)} \leqslant \kappa_1 \tag{12}$$

where κ_i $(i = 1, 2,...)$ denotes a constant, not necessarily always the same, independent of the indices n, k. For every fixed integer $k > k_0$ we have

$$\int_{\Omega_{k+1}} \{a_{ij} D_i u_n D_j v + p(u_n)\, v\} = \int_{\Omega_{k+1}} \{fv + f_i D_i v\}\, dx \tag{13}$$

for each $v \in W_0^{1,2}(\Omega_{k+1})$ whenever $n \geqslant k+1$. Therefore using the method to obtain the partially interior L^∞-estimate for an elliptic equation (cf. [8, Chap. 3, Sect. 13]) we see that there is a constant $\kappa(k)$ depending possibly on k such that for all $n \geqslant k+1$

$$\|u_n\|_{L^\infty(\Omega_k)} \leqslant \kappa(k). \tag{14}$$

We wish to mention that because $p(t)\, t \geqslant \lambda t^2 \geqslant 0$ $\forall t \in \mathbb{R}$, the term involving $p(u_n)$ in (13) will drop out and does not cause any difficulty; therefore the estimate process of [8] cited above can be repeated verbatim. Since for each $k > k_0$ the imbedding of $W^{1,2}(\Omega_k)$ into $L^2(\Omega_k)$ is compact, using a diagonal process we deduce from (12) that we can extract from $\{u_n\}_{n > k_0}$ a subsequence, still denoted by the same notation for convenience, such that as $n \to \infty$:

$$u_n \text{ converges weakly to } u \qquad \text{in} \quad W_0^{1,2}(\Omega),$$

$$u_n(x) \text{ converges to } u(x) \qquad \text{for a.a. } x \in \Omega.$$

Since $p(\cdot)$ is continuous, $p(u_n(x))$ converges to $p(u(x))$ for a.a. $x \in \Omega$. Then it follows from (14) that for every $k > k_0$, by the Lebesque convergence theorem, $p(u_n(x))$ converges to $p(u(x))$ weakly in $L^2(\Omega_k)$ as $n \to \infty$. Now given any $v \in W_0^{1,2}(\Omega)$ with compact support. We choose $k > k_0$, k sufficiently large so that support of $v \subset \Omega_{k+1}$. Then letting $n \to \infty$ in (13) we obtain

$$\int_{\Omega} \{a_{ij} D_i u D_j v + p(u)\, v\} dx = \int_{\Omega} \{fv + f_i D_i v\}\, dx \qquad \text{Q.E.D.}$$

We next consider the nonlinear BVP (1), (2) with

$$Au = -D_i a_i(x, u, \operatorname{grad} u) + a_0(x, u, \operatorname{grad} u)$$

and the functions $a_l (l = 0, 1, ..., N)$ satisfy the following conditions:

(A1) Each a_l is a function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and of Caratheodory's type: $a_l(x, \eta, \xi)$ is measurable in x for fixed $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and is continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$ for a.a. $x \in \Omega$. Moreover there exist a constant c_l and a function $k_l(x) \geqslant 0$ a.e. on Ω, $k_l(\cdot) \in L^2(\Omega)$ such that

$$|a_l(x, \eta, \xi)| \leqslant c_l(k_l(x) + |\eta| + |\xi|) \tag{15}$$

for a.a. $x \in \Omega$ and $\forall (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

(A2) For a.a. $x \in \Omega$, $\forall \eta \in \mathbb{R}$; $\xi, \xi' \in \mathbb{R}^N$

$$[a_i(x, \eta, \xi) - a_i(x, \eta, \xi')](\xi_i - \xi_i') > 0 \qquad \text{if} \quad \xi \neq \xi'. \tag{16}$$

(A3) There exists $\nu > 0$ such that for a.a. $x \in \Omega$, $\forall (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$a_i(x, \eta, \xi)\, \xi_i \geqslant \nu |\xi|^2. \tag{17}$$

We prove

THEOREM II. *Suppose that the function $p(\cdot)$ is continuously differentiable and satisfies the following conditions*

(P3) $p(0) = 0$ *and* $p'(t) \geqslant \lambda > c_0 + c_0^2/4\nu$ $\forall t \in \mathbb{R}$ *where* c_0 *and* ν *are the constants in* (15) *and* (17), *respectively.*

(P4) *There exist constants* $\alpha, \beta \geqslant 0$ *such that*

$$p'(t) \leqslant \alpha |p(t)| + \beta \qquad \forall t \in \mathbb{R}.$$

Suppose further that the operator A satisfies conditions (A1), (A2), *and* (A3) *above. Then given any* $f \in L^2(\Omega) \cap L^r_{\text{loc}}(\Omega)$ *with* $r > N/2$, $f_i \in L^2(\Omega) \cap L^4(\Omega) \cap L^s_{\text{loc}}(\Omega)$ *with* $s > N$ $(i = 1, ..., N)$ *the* BVP (1), (2) *has a solution* $u \in W_0^{1,2}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ *in the sense that* $p(u) \in L^\infty_{\text{loc}}(\Omega) \cap L^2(\Omega)$ *and for every* $v \in W_0^{1,2}(\Omega)$ *we have*

$$\int_{\Omega} \{a_i(x, \eta, \operatorname{grad} u)\, D_i v + a_0(x, \eta, \operatorname{grad} u)\, v + p(u)\, v\}\, dx$$

$$= \int_{\Omega} \{fv + f_i D_i v\}\, dx. \tag{18}$$

Before proving Theorem II we wish to make a few remarks:

Remark II. The results of Brézis and Browder in [2] and of Webb in [12] do not seem to apply to our equation. In fact, in [2, 12] the strongly nonlinear term p is allowed to depend on x *but subject to the condition* (cf. *condition* 2′), p. 590 in [2] and condition (G2), p. 125 of [12]):

> The equation $p_s(x) = \sup_{|t| \leqslant s} |p(x, t)|$ defines an $L^1(\Omega)$ function for $0 \leqslant s < \infty$.

Our function p does not satisfy this condition. On the other hand, [2, 12] include operators A of order $2m \geqslant 2$ and the right-hand side of Eq. (1) may be any element of the dual of $W_0^{m,2}(\Omega)$. But the solution u obtained is of a weaker sense: $p(u) \in L^1(\Omega)$ and (18) is only valid for $v \in W_0^{m,2}(\Omega) \cap L^\infty(\Omega)$.

Remark III. The linear operator in Eq. (3) is obviously a special case of the operator considered in Theorem II with $c_0 = 0$. Then condition (P3) is reduced to condition (P1).

Remark IV. As pointed out in Remark I, the function

$$p(t) = e^t - 1 \quad \text{if } t \geqslant 0, \quad p(t) = -e^{-t} + 1 \quad \text{if } t < 0$$

does not satisfy condition (P2) of [1]. However it satisfies (P4). Assuming that (P3) (and *a fortiori* (P1)) is satisfied, then elementary computations show that (P4) implies

(i) if $\alpha > 0$ then

$$p(t) \leqslant \delta(e^{\alpha t} - 1) \quad \text{if } t \geqslant 0, \quad p(t) \leqslant \delta(1 - e^{-\alpha t}) \quad \text{if } t \leqslant 0$$

where $\delta = \beta/\alpha$.

(ii) if $\alpha = 0$ then $|p(t)| \leqslant \beta |t|$ $\forall t \in \mathbb{R}$. Since (P3) implies $p(t) \geqslant \lambda t^2$, in this case using the well-known theory of coercive pseudo-monotone operators (cf., e.g., [10, Chap. 2, Sect. 2]) it can be seen that the BVP (1), (2) has a solution in $W_0^{1,2}(\Omega)$ for any $f, f_i \in L^2(\Omega)$ $(i = 1, ..., N)$.

Thus, in constrast to condition (P2), condition (P4) with $\alpha > 0$ still allows some exponential growths. On the other hand, the function

$$p(t) = e^{t^2} + t - 1 \quad \text{if } t \geqslant 0, \quad p(t) = -e^{t^2} + t + 1 \quad \text{if } t \leqslant 0,$$

while satisfying (P1), satisfies neither (P2) nor (P4).

Remark V. In general, condition (P2) implies condition (P4): hence by adopting a weaker version of the condition (P2) on $p(\cdot)$ imposed by Theorem 3.1 of [1], namely (P4), but at the cost of stricter requirement on the function f_i $(i=1,..., N)$ we obtain, even for nonlinear operators A, a solution which satisfies the BVP (1), (2) in a stronger sense than the one obtained in [1, Theorem 3.1]. In fact, assuming that (P1) is satisfied, if in (P2)

$$|p(\mu t)| \leqslant \gamma(\mu)|p(t)|$$

the function $\gamma(\cdot)$ is right-hand differentiable at 1 and $\gamma(1)=1$ (which we can assume without loss of generality because $p(\cdot)$ is increasing). Then for $\mu>1$, $t>0$ we have

$$p(\mu t)-p(t) \leqslant [\gamma(\mu)-1]\, p(t).$$

So for some $\theta \in (1, \mu)$ we have

$$p'(\theta t) \leqslant \frac{\gamma(\mu)-1}{\mu-1} \frac{p(t)}{t}.$$

Since $p'(0)$ exists, $p(t)/t$ is bounded for $t \in [0, 1]$. Thus letting $\mu \downarrow 1$ we obtain

$$p'(t) \leqslant D_+\gamma(1)\, p(t)+\kappa_1 \qquad \forall t \geqslant 0$$

for some constant κ_1. Similarly, it can be shown that for $t \leqslant 0$

$$p'(t) \leqslant D_+\gamma(1)|p(t)|+\kappa_1.$$

Proof of Theorem II. We use the notations introduced in the proof of Theorem I. By assumption (P3), $\exists \varepsilon^*>0$ such that

$$\bar{\varepsilon}:=\lambda-\left(c_0+\frac{c_0^2}{4(\nu-\varepsilon^*)}\right)>0.$$

Then because of the assumptions (A1) and (A3) elementary computations yield

$$\begin{aligned} a_i(x, \eta, \xi)\, \xi_i&+a_0(x, \eta, \xi)\, \eta+\lambda \eta^2 \\ &\geqslant \varepsilon(|\eta|^2+|\xi|^2)-\frac{c_0^2}{\varepsilon} k_0^2(x) \end{aligned} \tag{19}$$

for a.a. $x \in \Omega$, $\forall(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $2\varepsilon=$ minimum $(\bar{\varepsilon}, \varepsilon^*)$. By writing

$$Au+p(u)=-D_i a_i(x, u, \operatorname{grad} u)+a_0(x, u, \operatorname{grad} u)+\lambda u+[p(u)-\lambda u]$$

and noting that

$$[p(t)-\lambda t]\, t \geqslant 0 \qquad \forall t \in \mathbb{R},$$

we conclude from (19) and the result proved in [4] that for each integer $n > k_0$ there exists $u_n \in W_0^{1,2}(\Omega_n) \cap L^\infty(\Omega_n)$ such that

$$\int_{\Omega_n} \{a_i(x, u_n, \operatorname{grad} u_n)\, D_i v + a_0(x, u_n, \operatorname{grad} u_n)\, v + p(u_n)\, v\}\, dx$$

$$= \int_{\Omega_n} \{fv + f_i D_i v\}\, dx \tag{20}$$

for all $v \in W_0^{1,2}(\Omega_n)$. Taking $v = u_n$ in (20) we then obtain by using (19):

$$\|u_n\|_{W^{1,2}(\Omega_n)} \leqslant \kappa_2 \tag{21}$$

for some constant κ_2 independent of n.

We now obtain an L^2 estimate for the $p(u_n)$. Since $p(0) = 0$, $p(\cdot)$ is continuously differentiable and u_n is in $L^\infty(\Omega_n) \cap W_0^{1,2}(\Omega_n)$, we know ([7, Lemma 7.5, p. 144]) that $p(u_n)$ is in $W_0^{1,2}(\Omega_n)$ and

$$D_i p(u_n) = p'(u_n)\, D_i u_n \qquad (i = 1,..., N).$$

Thus, with $v = p(u_n)$, (20) yields

$$\int_{\Omega_n} \{p'(u_n)\, a_i(x, u_n, \operatorname{grad} u_n)\, D_i u_n + a_0(x, u_n, \operatorname{grad} u_n)\, p(u_n) + p(u_n)^2\}\, dx$$

$$= \int_{\Omega_n} \{fp(u_n) + p'(u_n)\, f_i D_i u_n\}\, dx. \tag{22}$$

By Hölder's inequality,

$$\left|\int_{\Omega_n} p'(u_n)\, f_i D_i u_n\, dx\right| \leqslant \frac{v}{4} \int_{\Omega_n} p'(u_n)\, |\operatorname{grad} u_n|^2\, dx$$

$$+ \frac{1}{v} \int_{\Omega_n} p'(u_n)\, F^2(x)\, dx$$

where $F^2(x) = f_1^2(x) + \cdots + f_N^2(x)$. Therefore using (17) we obtain from (22)

$$\int_{\Omega_n} a_0(x, u_n, \operatorname{grad} u_n)\, p(u_n)\, dx + \int_{\Omega_n} p(u_n)^2\, dx$$

$$\leqslant \int_{\Omega_n} fp(u_n)\, dx + \frac{1}{v} \int_{\Omega_n} p'(u_n)\, F^2(x)\, dx. \tag{23}$$

By assumption (P4) we have

$$\begin{aligned}
&\frac{1}{\nu}\int_{\Omega_n} p'(u_n)\,F^2(x)\,dx \\
&\quad\leqslant \frac{1}{\nu}\left\{\alpha\int_{\Omega_n} |p(u_n)|\,F^2\,dx+\beta\int_{\Omega_n} F^2\,dx\right\} \\
&\quad\leqslant \int\frac{1}{\nu}\left\{\alpha\left[\frac{\nu}{16\alpha}\int_{\Omega_n}|p(u_n)|^2\,dx+\frac{4\alpha}{\nu}\int_{\Omega_n}F^4\,dx\right]+\beta\int_{\Omega_n}F^2\,dx\right\} \\
&\quad\leqslant \frac{1}{16}\int_{\Omega_n}|p(u_n)|^2\,dx+\frac{4\alpha^2}{\nu^2}\int_{\Omega_n}F^4\,dx+\frac{\beta}{\nu}\int_{\Omega_n}F^2\,dx. \qquad (24)
\end{aligned}$$

By assumption (A1) we also have

$$\begin{aligned}
&\left|\int_{\Omega_n} a_0(x, u_n, \operatorname{grad} u_n)\,p(u_n)\,dx\right| \\
&\qquad\leqslant c_0\int_{\Omega_n}\{k_0(x)+|u_n|+|\operatorname{grad} u_n|\}\,|p(u_n)|\,dx.
\end{aligned}$$

Since $k_0(\cdot)\in L^2(\Omega)$, using (21) we deduce from the last inequality that

$$\left|\int_{\Omega_n} a_0(x, u_n, \operatorname{grad} u_n)\,p(u_n)\,dx\right|\leqslant\tfrac{1}{16}\int_{\Omega_n}p(u_n)^2\,dx+\kappa_3 \qquad (25)$$

for some constant κ_3 independent of n. Finally we also have

$$\left|\int_{\Omega_n} fp(u_n)\right|dx\leqslant\tfrac{1}{16}\int_{\Omega_n}p(u_n)^2\,dx+4\int_{\Omega_n}f^2\,dx. \qquad (26)$$

Since f^2, F^2, $F^4\in L^1(\Omega)$ by hypothesis, we deduce from (23), (24), (25), and (26) that

$$\|p(u_n)\|_{L^2(\Omega_n)}\leqslant\kappa_4 \qquad (27)$$

for some constant κ_4 independent of n. We next extend u_n to the whole of Ω by defining $u_n(x)\equiv 0$ for $x\notin\Omega_n$ and, for convenience, still denote by u_n this function of $W_0^{1,2}(\Omega)$.

We are now ready to pass to the limit. As in the proof of Theorem I, it follows from (21) that we can extract from $\{u_n\}$ a subsequence, which we still denote by $\{u_n\}$ for convenience, such that

$\{u_n\}$ converges strongly to u in $L^2(\Omega_k)$ for each integer $k > k_0$,

$\{u_n\}$ converges weakly to u in $W_0^{1,2}(\Omega)$,

$\{u_n\}$ converges almost everywhere to u on Ω,

$\{p(u_n)\}$ converges almost everywhere to $p(u)$ on Ω.

Then, since $\{p(u_n)\}$ is bounded in $L^2(\Omega)$ by (27), we deduce that $\{p(u_n)\}$ converges weakly to $p(u)$ in $L^2(\Omega)$. In fact, let $w(\cdot) \in L^2(\Omega)$ and, given any $\varepsilon' > 0$, let $k > k_0$, k chosen sufficiently large so that with the constant κ_4 in (27) we have

$$\kappa_4 \left(\int_{\Omega - \Omega_k} w^2 \, dx \right)^{1/2} < \frac{\varepsilon'}{4}.$$

Furthermore, since $\{p(u_n)\}$ converges weakly to $p(u)$ in $L^2(\Omega_k)$ (cf. e.g., [10, Lemma 1.3, p. 12]) we can choose an integer L suffiently large so that

$$\left| \int_{\Omega_k} \{p(u_n) - p(u)\} \, w \, dx \right| < \frac{\varepsilon'}{2} \qquad \forall n > L.$$

The last two inequalities then yield

$$\left| \int_{\Omega} \{p(u_n) - p(u)\} \, w \, dx \right| < \varepsilon' \quad \forall n > L.$$

Next we recall that for each integer $k > k_0$ we have constructed in the proof of Theorem I a function $\zeta_k(\cdot) \in C_0^1(\Omega)$ with

$$\zeta_k(x) \equiv 1 \text{ if } x \in \Omega_k,\ \zeta_k(x) = 0 \text{ if } x \notin \Omega_{k+1},\ 0 \leqslant \zeta_k(x) \leqslant 1\ \forall x \in \Omega,$$

$$|\text{grad } \zeta_k(x)| \leqslant \kappa_1\ \forall x \in \Omega \text{ for some constant } \kappa_1 \text{ independent of } k.$$

We shall next show that

$$\limsup_n \langle (A + \lambda)\, u_n, u_n - u \rangle \leqslant 0,$$

where $\langle \cdot, \cdot \rangle$ *stands for the pairing between* $W_0^{1,2}(\Omega)$ *and its dual.* Let $\varepsilon' > 0$ be arbitrarily given. Since $(1 - \zeta_k)\, u$ converges strongly in $W_0^{1,2}(\Omega)$ to 0 as $k \to \infty$ and the sequences $\{a_l(x, u_n, \text{grad } u_n)\}$, $l = 0, 1, \ldots, N$, are bounded in $L^2(\Omega)$ because the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ by (21), for all sufficiently large k we have

$$|\langle (A + \lambda)\, u_n, (1 - \zeta_k)\, u \rangle| < \varepsilon'. \tag{28}$$

Similarly, for all sufficiently large k we have

$$|\langle f, (1-\zeta_k)\, u_n\rangle| = \left|\int_\Omega f(1-\zeta_k)\, u_n\, x\right| < \varepsilon' \qquad \forall n > k_0. \tag{29}$$

Furthermore

$$\langle -D_i f_i, (1-\zeta_k)\, u_n\rangle = \int_\Omega \{f_i(1-\zeta_k)\, D_i u_n - f_i \cdot D_i \zeta_k \cdot u_n\}\, dx.$$

Since $|\mathrm{grad}\, \zeta_k(\cdot)|$ is bounded on Ω by a number independent of k and is nonzero only for $k \leqslant |x| \leqslant k+1$ and because $f_i \in L^2(\Omega)$, $i = 1,..., N$ and $\{u_n\}$ is bounded in $W_0^{1,2}(G)$ we deduce from the last equation that for all sufficiently large k we have

$$|\langle -D_i f_i, (1-\zeta_k)\, u_n\rangle| < \varepsilon' \qquad \forall n > k_0. \tag{30}$$

We now fix a k, $k > k_0$ such that (28), (29), and (30) are simultaneously satisfied. Taking $v = (1-\zeta_k)\, u_n$ in (20) we obtain

$$\begin{aligned} &\langle (A+\lambda)\, u_n, (1-\zeta_k)\, u_n\rangle + \int_\Omega [p(u_n) - \lambda u_n](1-\zeta_k)\, u_n\, dx \\ &\quad = \langle f - D_i f_i, (1-\zeta_k)\, u_n\rangle. \end{aligned} \tag{31}$$

Since $[p(t) - \lambda t]\, t \geqslant 0$ $\forall t \in \mathbb{R}$, we deduce from (29), (30), and (31) that

$$\langle (A+\lambda)\, u_n, (1-\zeta_k)\, u_n\rangle < 2\varepsilon' \qquad \forall n \geqslant k_0.$$

This inequality and (28) yield

$$\langle (A+\lambda)\, u_n, (1-\zeta_k)(u_n - u)\rangle < 3\varepsilon'. \tag{32}$$

Furthermore, we obtain from (20) with $v = \zeta_k(u_n - u)$,

$$\begin{aligned} &\langle (A+\lambda)\, u_n, \zeta_k(u_n-u)\rangle + \int_\Omega [p(u_n) - \lambda u_n]\, \zeta_k(u_n-u)\, dx \\ &\quad = \langle f - D_i f_i, \zeta_k(u_n - u)\rangle \end{aligned}$$

if $n \geqslant k+1$. Since the sequence $\{p(u_n) - \lambda u_n\}$ is bounded in $L^2(\Omega)$ by (21) and (27) and because $\{\zeta_k u_n\}_n$ converges strongly to $\zeta_k u$ in $L^2(\Omega)$ and weakly to $\zeta_k u$ in $W_0^{1,2}(\Omega)$ we deduce that there exists an integer L such that

$$|\langle A+\lambda)\, u_n, \zeta_k(u_n - u)\rangle < \varepsilon' \qquad \text{if} \quad n > L. \tag{33}$$

From (32) and (33) we conclude that

$$\langle (A+\lambda)\, u_n, u_n - u\rangle < 4\varepsilon' \qquad \text{if} \quad n > L.$$

Since $\varepsilon' > 0$ is arbitrary we therefore have

$$\limsup_n \langle (A+\lambda)\, u_n, u_n - u\rangle \leqslant 0. \tag{34}$$

On the other hand, the assumptions (A1), (A2) together with the inequality (19) imply that the operator $A+\lambda$ from $W_0^{1,2}(\Omega)$ into its dual is pseudo-monotone ([3], see also [9]), we therefore conclude from (34) that $\{Au_n\}$ converges weakly to Au in the dual $W^{-1,2}(\Omega)$ of $W_0^{1,2}(\Omega)$. Now let $w \in W_0^{1,2}(\Omega)$ be arbitrarily given. Replace v by $\zeta_k w$ in (20) with $k \leqslant n-1$ and then letting $n \to \infty$ we obtain

$$\langle Au, \zeta_k w\rangle + \int_\Omega p(u)\, \zeta_k w\, dx = \int_\Omega \{ f\zeta_k w + f_i D_i(\zeta_k w)\}\, dx.$$

Finally letting $k \to \infty$, we conclude that

$$\langle Au, w\rangle + \int_\Omega p(u)\, w\, dx = \int_\Omega \{ fw + f_i D_i w\}\, dx.$$

It remains to show that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. For this purpose let us return once more to the sequence $\{u_n\}_{n>k_0}$ defined by (20). It is not difficult to see that the method of obtaining a partially interior L^∞-estimate for the solution of an elliptic equation as described in [8, Chap. 3, Sect. 13] can be repeated verbatim for (20). Therefore for every k, $k > k_0$ there is a constant $\kappa(k)$ depending possibly on k but independent of n such that for all $n \geqslant k+1$

$$\|u_n\|_{L^\infty(\Omega_k)} \leqslant \kappa(k).$$

Since the solution u that we obtained is the pointwise limit a.e. of $\{u_n\}_{n>k}$, we deduce that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Q.E.D.

We also have the following variation of Theorem II:

THEOREM III. *Suppose that the function $p(\cdot)$ and the operator A are like in Theorem* II. *Then given any $f \in L^2(\Omega) \cap L^r_{\mathrm{loc}}(\Omega)$ with $r > \frac{N}{2}$, $f_i \in L^2(\Omega) \cap L^\infty(\Omega)$ $(i = 1,..., N)$, the BVP* (1), (2) *has a solution $u \in W_0^{1,2}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ in the sense of Theorem* II.

Proof. The proof is similar to that of Theorem II with the following modification in the computation leading to the L^2 estimate (27) for the

$p(u_n)$: From (22) taking into account the ellipticity (17) of A as well as the fact that $p'(t)>0\ \forall t\in\mathbb{R}$, we obtain

$$\int_{\Omega_n} a_0(x, u_n, \operatorname{grad} u_n)\, p(u_n)\, dx + \int_{\Omega_n} p(u_n)^2\, dx$$

$$\leqslant \int_{\Omega_n} fp(u_n)\, dx + \int_{\Omega_n} f_i p'(u_n)\, D_i u_n\, dx. \tag{35}$$

By assumption (P4) we have

$$\int_{\Omega_n} |f_i p'(u_n)\, D_i u_n|\, dx \leqslant \alpha \int_{\Omega_n} |f_i|\, |p(u_n)|\, |D_i u_n|\, dx + \beta \int_{\Omega_n} |f_i|\, |D_i u_n|\, dx$$

$$\leqslant \kappa_1 \int_{\Omega_n} |p(u_n)|\, |\operatorname{grad} u_n|\, dx + \kappa_2$$

$$\leqslant \varepsilon \int_{\Omega_n} p(u_n)^2\, dx + \kappa_3(\varepsilon)$$

for any $\varepsilon>0$ given, where $\kappa_3(\varepsilon)$ is a constant depending on ε. The other integrals in (35) are estimated as in the proof of Theorem II. Q.E.D.

Theorems II and III do not apply, for example, to the function

$$p(t)=e^{t^2}+t-1 \quad \text{if } t\geqslant 0,\ p(t)=-e^{t^2}+t+1 \text{ if } t\leqslant 0$$

which does not satisfy condition (P4). For such functions we prove

THEOREM IV. *Suppose that the function $p(\cdot)$ is continuously differentiable and satisfies both condition* (P3) *of Theorem* II *and the following condition*:

(P5) *There exist constants $\gamma_1, \gamma_2 \geqslant 0$ such that*

$$p'(t) \leqslant \gamma_1 p(t)^2 + \gamma_2 \qquad \forall t\in\mathbb{R}.$$

Suppose also that the operator A satisfies conditions (A1), (A2), *and* (A3). *Then given any $f\in L^2(\Omega)\cap L^r_{\mathrm{loc}}(\Omega)$ with $r>N/2$ and $f_i\in L^2(\Omega)\cap L^\infty(\Omega)$ $(i=1,\ldots,N)$ such that*

$$\frac{\gamma_1}{4\nu}\left\|\sum_{i=1}^N f_i^2\right\|_{L^\infty(\Omega)} < 1,$$

where ν is the ellipticity constant of A in (17), *the* BVP (1), (2) *has a solution $u\in W_0^{1,2}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. Again, the proof is similar to the proof of Theorem II with a slight modification in the computation leading to the L^2-estimate (27) for the $p(u_n)$: In (22) we estimate the last integral as follows

$$\int_{\Omega_n} |p'(u_n) f_i D_i u_n| \, dx \leqslant \nu \int_{\Omega_n} p'(u_n) |\text{grad } u_n|^2 \, dx + \frac{1}{4\nu} \int_{\Omega_n} \left(\sum_{i=1}^{N} f_i^2 \right) p'(u_n) \, dx. \qquad \text{Q.E.D.}$$

To deal with the case when we only have $p'(t) \geqslant 0$ $\forall t \in \mathbb{R}$ we have to introduce some weighted Sobolev's spaces. We *suppose for the rest of the paper that* $0 \notin \bar{\Omega}$. For a number $\tau \in \mathbb{R}$ we denote by $L^2(\Omega, |x|^\tau)$ the Banach space of (equivalence classes of) functions u such that

$$\|u\|_{L^2(\Omega, |x|^\tau)} = \left(\int_\Omega u^2 |x|^\tau \, dx \right)^{1/2} < \infty$$

equipped with the norm $\|\cdot\|_{L^2(\Omega, |x|^\tau)}$. By $\Gamma^{1,2}(\Omega, |x|^\tau, 1)$ we denote the Banach space of distributions u on Ω such that

$$u \in L^2(\Omega, |x|^\tau), \qquad D_i u \in L^2(\Omega) \qquad (i = 1, ..., N)$$

and equipped with the norm

$$\|u\|_{\Gamma^{1,2}(\Omega, |x|^\tau, 1)} = \left\{ \|u\|^2_{L^2(\Omega, |x|^\tau)} + \sum_{i=1}^{N} \|D_i u\|^2_{L^2(\Omega)} \right\}^{1/2},$$

$\mathring{\Gamma}^{1,2}(\Omega, |x|^\tau, 1)$ denotes the closure of $C_0^\infty(\Omega)$ in $\Gamma^{1,2}(\Omega, |x|^\tau, 1)$. We have the following counterpart of Theorem I.

THEOREM V. *Suppose that*

(P6) $p(0) = 0$ *and* $p'(t) \geqslant 0$ $\forall t \in \mathbb{R}$. *Then for either* $\tau = 2 \neq N$ *or* $\tau > 2$, *given* $f \in L^2(\Omega, |x|^\tau) \cap L^r_{\text{loc}}(\Omega)$ *with* $r > N/2$ *and* $f_i \in L^2(\Omega) \cap L^s_{\text{loc}}(\Omega)$ *with* $s > N$ $(i = 1, ..., N)$, *the BVP* (3), (4) *has a unique solution* $u \in \mathring{\Gamma}^{1,2}(\Omega, |x|^{-\tau}, 1) \cap L^\infty_{\text{loc}}(\Omega)$ *in the sense that for every* $v \in \mathring{\Gamma}^{1,2}(\Omega, |x|^{-\tau}, 1)$ *with compact support we have* (5).

Proof. The proof is similar to that of Theorem I, except that to arrive at the counterpart of (12) we now have to make use of the following

LEMMA (cf. [1, Theorem 1.3; 13, Theorem 1.1]). *If either* $\tau = 2$ *and* $N \neq 2$ *or* $\tau > 2$, *and* $0 \notin \bar{\Omega}$ *then there exists a constant* K *such that*

$$\int_\Omega |u(x)|^2 |x|^{-\tau} \, dx \leqslant K \int_\Omega |\text{grad } u(x)|^2 \, dx \quad \forall u \in C_0^\infty(\Omega).$$

Remark VI. Remark I with obvious modifications applies to Theorem V as well.

Concluding Remark. Strongly nonlinear elliptic boundary value problems for unbounded domains are also extensively discussed in [11] where the differential operator may be of higher order $2m \geqslant 2$ and the strongly nonlinear term(s) may be more general in that they may depend on derivatives of order $\leqslant m$. Thus Theorem 5 in [6] can be deduced from Satz 5 in [11] although the proof are different. However, in [11] the right-hand side is a function on Ω instead of an element of the dual of $W_0^{1,2}(\Omega)$ as we allow in this paper. Furthermore in [11] a solution is obtained in the sense that (5), e.g., is valid for all $v \in C_0^\infty(\Omega)$ instead of $v \in W^{1,2}(\Omega)$ with compact support as we prove. We have used this latter fact in an essential way to prove the uniqueness of the solution in Theorem I. Of course to obtain that fact we have had to put some additional but rather mild restrictions on the f and f_i's. We note that under appropriate extra assumptions on the nonlinear differential operator, the solution in Theorem II can also be proved to be unique by the same technique as in Theorem I.

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References

1. V. Benci and D. Fortunato, Some nonlinear elliptic problems with asymptotic conditions, *Nonlinear Anal.* **3** (1979), 157–174.
2. H. Brézis and F. E. Browder, Strongly nonlinear elliptic boundary value problems, *Ann. Scuola Norm. Sup. Pisa* **5** (1978), 587–603.
3. F. E. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), 2659–2661.
4. N. P. Các, On bounded solutions of a strongly nonlinear elliptic equation, *Pacific J. Math.* **57** (1975), 53–58.
5. N. P. Các, Nonlinear elliptic boundary value problems for unbounded domains, *J. Differential Equations* **45** (1982), 191–198.
6. N. P. Các, On some quasilinear elliptic boundary value problems with conditions at infinity, *J. Differential Equations* **52** (1984), 342–355.
7. D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer–Verlag, New York, 1977.
8. O. A. Ladyzhenskaya and N. N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Academic Press, New York, 1968.
9. R. Landes and V. Mustonen, On pseudo-monotone operators and nonlinear noncoercive variational problems on unbounded domains, *Math. Ann.* **248** (1980), 241–246.

10. J. L. LIONS, "Quelques méthodes de résolution des problèms aux limites non linéaires," Dunod, Paris, 1969.
11. C. G. SIMADER, Über schwache Lösungen des Dirichletproblems für streng nichtlineare elliptische Differentialgleichungen, *Math. Z.* **150** (1976), 1–26.
12. J. R. L. WEBB, Boundary value problems for strongly nonlinear elliptic equations, *J. London Math. Soc.* (2) **21** (1980), 123–132.
13. V. BENCI AND D. FORTUNATO, Weighted Sobolev spaces and the nonlinear Dirichlet problem in unbounded domains, *Ann. Mat. Pura Appl.* **71** (1979), 319–336.